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A Preference-based Interpretation of AHP

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The relationship between MAUT and AHP for three-level hierarchic structures is demonstrated based on a common framework interpreting multiattribute decision making techniques. A theorem showing the conditions under which two multiattribute decision making techniques result in a consistent preference structure is proved. It can be justified based on this theorem that the rank reversal problem in AHP resulting from the addition or deletion of alternatives is caused by multiplying inappropriate criteria priorities with local priorities for alternatives. A scaling method, AHP′, is introduced synthesizing judgments from pairwise comparisons within and among criteria into value scales in MAUT.

Key words—Analytic Hierarchy Process, Multiattribute Utility Theory, rank reversal, meanings of weights, scaling methods

INTRODUCTION

EVER SINCE THE Analytic Hierarchy Process (AHP) was introduced by [15], there has been a wide discussion about the empirical effectiveness and theoretical validity of the technique [2–4, 6, 7, 16–18, 23]. The discussion has focused on four areas: the axiomatic foundation, the correct meanings of weights or priorities for attributes or criteria, the 1–9 measurement scale, and the rank reversal problem due to addition or deletion of alternatives [6]. This paper focuses instead on the priorities among alternatives.

With the possible exception of the 1–9 measurement scale, the other areas have, in our view, been partially resolved, at least for three-level hierarchic structures. Saaty has given a sound, but incomplete axiomatic foundation for AHP because it focuses mostly on paired comparisons among alternatives within criteria, while the interdependence between alternatives and criteria and among criteria across levels remains ambiguous [16]. The meaning of priorities for criteria (or attribute weights) in the AHP has been given explicitly in terms of preferences [13, 22]. A consensus among critical AHP investigators has emerged that the rank reversal problem is caused by the ignorance of the relationship between the definitions of weights and the associated attribute scales [12, 19]. Strictly speaking, all four areas are closely related in that a resolution for one area can be approached indirectly through that for another. For example, if the meanings of criteria priorities are defined in accordance with the scales measuring the criteria, the rank reversal phenomena do not occur [12, 19]. These meanings would provide insights into developing measurement scales different from the 1–9 scale used in AHP.

An important issue fundamental to all four areas, but slighted in the literature, is the meanings of the ratios or entries in the pairwise comparison matrices of alternatives with respect to criteria [1, 8, 13, 24]. Even Saaty treated these entries as primitive scales without explicit, concrete meanings [16]. Others use objective or subjective scores in their expositions of interpreting the meanings of priorities or resolving the rank reversal problem [3, 4]. These scores do not exist in most practical situations, limiting the signififcance of these theoretically justified expositions from an operational view point. The interpretation of the meanings of entries in pairwise comparison matrices of alternatives can be helpful not only in dispelling confusion in debate but also in relating AHP to other theoretically sound techniques, such as multiattribute utility theory, or MAUT [10].

In this paper, we interpret the meanings of the entries in pairwise comparison matrices in terms of the decision maker's preference judgments and introduce a modified version of AHP, AHP′, based on that interpretation. We first provide a common framework for comparing multiattribute decision making techniques. We then develop a theorem with the necessary and sufficient condition, such that two additive preference aggregation, or decision rules result in consistent preference structures. The meanings of criteria priorities in AHP, or attribute weights in MAUT, and a solution for the rank reversal problem are given based on the theorem. Finally, the scaling method of AHP′ is illustrated.

A FRAMEWORK

In attempting to interpret the meanings of priorities for criteria and solving the rank reversal problem in AHP, theorists have used different perspectives. Confusion may thus arise not because the logic they developed is inconsistent, but because the languages they used make the concepts elusive [3–7, 20]. A common framework is necessary to provide a language for dispelling the confusion. Such a language was given in [13] for deductively comparing five multiattribute decision making techniques, including MAUT and AHP. We elaborate that language here as a basis for the development in the following sections.

First, we define some specific terms. An attribute space is defined as a collection of vectors in which each point (or vector) represents an alternative, the dimensionality of the space being composed of the attributes (or criteria) describing the alternative. Dimensionality means not only the number, but also the attributes themselves that constitute a space. Briefly speaking, a scale is a function mapping an empirical relational structure to a numerical one. "A *relational structure* is a set together with one or more relations on that set" [11]. The original measures (scales) for the attributes, either in natural or constructed units, are referred to as attribute levels. In this framework, all measurements are, in essence, relative in that the measurements for attributes or alternatives in one attribute space can be transformed directly or indirectly to those in another space. The notion of the original scales is defined for convenience to refer to the scales most frequently used. The same alternative can thus be represented by different sets of coordinates in different attribute spaces and, therefore, transformed from one attribute space to another by a multidimensional transformation function. The composition rule in AHP to derive composite priorities is a common example of a linear multidimensional transformation function. A unidimensional transformation function maps, on the other hand, one attribute from one scale to another. A decision rule in an attribute space is an arithmetic computation mapping multidimensional scales (either original or derived) into a unidimensional scale based on which relative worths of alternatives are determined. The additive multiattribute utility function in MAUT is, for example, a decision rule.

Scores refer specifically to dimensionless scales other than the original scales of natural units. Trade-offs among the attributes in an attribute space are the ratios of the relative contributions of attributes in terms of the scales specified in that space to a unit, prescribed by the corresponding decision rule, of the overall measure of an alternative. Mathematically, the trade-offs are the marginal rates of substitution of attribute scales relative to a constant amount of the overall measure of the alternative. The determination of weights (coefficients in an additive decision rule associated with attributes) implies, therefore, the trade-offs among the attributes in a particular attribute space. Though weights and priorities are conceptually different fundamental constructs in MAUT and AHP, they refer mathematically to the coefficients in the additive decision rules. Weights and priorities, as well as attributes and criteria, are, therefore, used interchangeably here.

More specifically, consider n attributes. Let $\mathbf{X}_i^t = (x_{i1}^t, x_{i2}^t, x_{i3}^t, \ldots, x_{in}^t)$ be alternative i represented in terms of the original scales in the original attribute space $\mathbb{T}$, for $i = 1, 2, 3, \ldots, m$, where x^t_{ij} is the attribute level of attribute j for alternative i. Let $\mathbb{U}$, $\mathbb{R}$ be two other attribute spaces with the same dimensionality (composed of the same set of attributes), but different attribute scales. Let V^u_j and $V^r_j, j = 1, 2, \ldots, n$, be the unidimensional transformation functions mapping the scale for attribute j from the original attribute space $\mathbb{T}$ into the scales in $\mathbb{U}$ and $\mathbb{R}$ respectively. Thus, $\mathbf{V}^u_i = (V^u_1(x^t_{i1}), V^u_2(x^t_{i2}), \ldots, V^u_n(x^t_{in}))$ and $\mathbf{V}^r_i = (V^r_1(x^t_{i1}), V^r_2(x^t_{i2}), \ldots, V^r_n(x^t_{in}))$ are two score vectors representing the same alternative i in spaces $\mathbb{U}$ and $\mathbb{R}$ respectively. Let Z^u and Z^r be two additive decision rules in spaces $\mathbb{U}$ and $\mathbb{R}$ of the forms shown in (1) and (2).

$$Z^u(x^t_{ij}) = \sum_{j=1}^{n} w^u_j V^u_j(x^t_{ij}) \tag{1}$$

and

$$Z^r(x^t_{ij}) = \sum_{j=1}^{n} w^r_j V^r_j(x^t_{ij}), \text{ for } i = 1, 2, 3, \ldots, m, \tag{2}$$

where w^u_j and w^r_j are the coefficients associated with attributes in the specified scales and thus the marginal rates of substitution of the attribute scales relative to constant amounts of the overall measures of an alternative in spaces $\mathbb{U}$ and $\mathbb{R}$ respectively. w^u_js and w^r_js are usually called weights or priorities and normalized so that the sums are equal to unity.

The relative worths and thus the ranking of a set of alternatives determined by a decision rule in one attribute space should be consistent with that in another, given that the dimensionalities, and thus the attributes, of the two attribute spaces are the same and that the decision maker's preference structures with bounded value functions are stable over time. Otherwise, one or both of the two decision rules are not indicative of the decision maker's preference structures. A preference structure is a mathematical construct representing the decision maker's preference relations among alternatives. The interesting question is: under what conditions do the two additive decision rules, Z^u and Z^r, defined by (1) and (2) in two attribute spaces, yield a consistent preference structure? The concept of *permissible transformation* is helpful in deriving the condition.

A permissible transformation is a mapping of a set from one scale to another in which the properties of an empirical relational structure are retained. In the context of preference structures for multiattribute alternatives, a permissible transformation implies that the relative worths among alternatives remain the same *before* and *after* the transformation. By definition, two types of such transformations exist. According to [11], a scale Z whose permissible transformations are

$$T(Z) = \alpha Z, \alpha > 0, \tag{3}$$

is called a ratio scale, where T is a transformation function. A scale Z whose permissible transformations are

$$T(Z) = \alpha Z + \beta, \alpha > 0 \text{ and } \beta \neq 0, \tag{4}$$

is called an interval scale. Note that for a ratio scale, the ratios of scale values (relative worths) are determined uniquely and that for an interval scale, the ratios of intervals are invariant. Two scales are mutually permissibly transformable if one of them can be transformed to the other according to either (3) or (4).

The necessary and sufficient conditions for mutually permissible transformation between additive, multiattribute decision rules is given in Appendix 1 as Theorem 1. More specifically, two additive, multiattribute decision rules in two spaces composed of the same set of attributes are mutually permissibly transformable if and only if the weights of one decision rule are proportional to the partial derivatives of the other decision rule with respect to the associated unidimensional value functions respectively, the proportionality being a constant.

RANK REVERSAL AND SOLUTIONS

Theorem 1 provides a basis for solving the rank reversal problem and defining the meaning of priorities in AHP in the context of an AHP-MAUT relationship. Both AHP and MAUT have sound axiomatic foundations. Though comparative study of the relationship between the two techniques has been done, no consensus has been reached [1, 8, 13]. We describe this relationship here using the language proposed.

The rank reversal problem and the meaning of weights are dual problems in that solving one problem gives a solution to the other. We argue that the rank reversal problem in AHP is caused by multiplying the normalized alternative priorities in the AHP scales by the attribute weights in scales other than the AHP scales and that the weight (priority) for an attribute (criterion) in relation to the AHP scales has a meaning different from that in the MAUT scales. In particular, the AHP weight for attribute j is the sum of the weighted values over all alternatives divided by the total of these sums for all attributes in the MAUT scales.[1] We use the term values to indicate specifically the scores measured in the MAUT scales in space $\mathbb{M}$.

To justify the two arguments, we need to define explicitly the meanings of the entries in pairwise comparison matrices for AHP based on the decision maker's preference judgments and then derive the relationship between AHP and MAUT, both leading to consistent preference structures. Consider a three-level hierarchy. That is, except for the focus and alternative levels, there is only one level of attributes. For m alternatives and n attributes, let x_{ij}^t indicate the level of attribute j for alternative i in the original attribute space $\mathbb{T}$. That is, $\mathbf{X}_i^t = (x_{i1}^t, x_{i2}^t, x_{i3}^t, \ldots, x_{in}^t)$ is a point (vector) of alternative i in $\mathbb{T}$. Suppose the MAUT scale for measuring x_{ij}^t is represented by the transformation function, V_j^m, for attribute j in space $\mathbb{M}$. If these transformation functions are bounded, as is usually the case, the trade-off ratios between attribute scores or values, $V_j^m(x_{ij}^t)$, must be taken into account by assigning weights to these attributes. Therefore, attribute weights already implicitly determine these trade-off ratios. Let the weight associated with attribute j be w_j^m. The total weighted scores (values) for alternative i in the MAUT scales are

$$Z^m(x_{ij}^t) = \sum_{j=1}^{n} w_j^m V_j^m(x_{ij}^t), \quad \text{for } i = 1, 2, \ldots, m, \tag{5}$$

where

Z^m = the multidimensional additive decision rule for MAUT in space $\mathbb{M}$,

w_j^m = the attribute weight in relation to the MAUT scales for attribute j,

x_{ij}^t = the attribute level measured in the original scales for alternative i with respect to attribute j in space $\mathbb{T}$,

V_j^m = the unidimensional transformation function sending attribute j from space $\mathbb{T}$ to space $\mathbb{M}$.

[1] A more concise version of the justification is demonstrated in [13].

In AHP, the ratios of the pairwise comparison matrix between alternatives for attribute j are, by definition, the ratios of the corresponding weighted values, $w_j^m V_j^m(x_{ij}^t)$. The weights w_j^m are cancelled out in the numerator and denominator and the remaining elements in the matrix become the ratios of the values $V_j^m(x_{ij}^t)$. The eigenvector approach in AHP results in priorities summing to unity, which is a scaling operator sending the measurement for an attribute from space $\mathbb{M}$ to space $\mathbb{A}$. The score of attribute j for alternative i in the AHP scales in space $\mathbb{A}$ is, assuming the matrix is perfectly consistent,

$$V_j^a(x_{ij}^t) = T_j \bigcirc V_j^m(x_{ij}^t) = \frac{V_j^m(x_{ij}^t)}{\sum_{k=1}^{m} V_j^m(x_{kj}^t)},$$

$$\text{for } i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n, \tag{6}$$

where V_j^a and T_j are the unidimensional transformation functions sending the measurement for attribute j from space $\mathbb{T}$ to space $\mathbb{A}$ and from space $\mathbb{M}$ to space $\mathbb{A}$ respectively. T_j is a normalization operator represented by the eigenvector approach. V_j^a is thus a two-step transformation; it first transforms x_{ij}^t to $V_j^m(x_{ij}^t)$ from $\mathbb{T}$ to $\mathbb{M}$, and then V_j^m to V_j^a from $\mathbb{M}$ to $\mathbb{A}$ by T_j.

Note that the denominator in (6),

$$\sum_{k=1}^{m} V_j^m(x_{kj}^t),$$

is a constant specifically for attribute j, given m alternatives. If we multiply the weights w_j^m in the MAUT scales with the scores $V_j^a(x_{ij}^t)$ derived in the AHP scales as usually done in the rank reversal problem literature, the relative worths among the alternatives will not be the same as those from (5) simply because the trade-off ratios between the attributes in the MAUT scales are changed. Rank reversal among alternatives may then occur. The mathematical exposition in Appendix 2 shows how the rank reversal problem can thus occur and how that problem can be resolved resulting in the correct meanings of weights and criteria priorities in MAUT and AHP.

Appendix 2 shows that the meaning of w_j^a in (13) results from the eigenvector approach to derive the normalized local priorities for alternatives and criteria, or attribute values and weights, in the AHP scales. w_j^a is also alternative-dependent because the denominator in (13) depends on the alternatives being considered.

As argued by some theorists [23], such meaning as defined in (13) is so complicated that the decision maker is not likely to be able to express it in any appropriate way. Some alternative scaling methods have been proposed to make that meaning more comprehensible to the decision maker while maintaining relative worths among alternatives [2, 21]. Others focus on resolving the rank reversal problem based on restrictive assumptions [9]. Whatever the scaling method, the condition in Theorem 1 must hold. In the next section, we introduce one such scaling method, called the AHP′ technique, for three-level hierarchic structures based on the transformation between the AHP and MAUT scales as justified in Appendix 2.

A NEW SCALING METHOD

The scaling method of AHP′ is similar to that of the B–G modified AHP originally introduced by Belton and Gear [2] and named in [18]. AHP′ is, however, different from the B–G modified AHP in that the meanings of ratios of pairwise comparison matrices are explicitly defined as preferences according to (6), while the B–G modified AHP only uses hypothetical, absolute scores (such as unbounded value functions) with no concrete operational implications as a means to illustrate the scaling method. It is not clear yet how the B–G modified AHP can be used for eliciting subjective judgments. AHP′ can therefore be operationalized based on the meanings of the entries in pairwise comparison matrices. In short, AHP′ requires the decision maker to make ratio judgments between alternatives and between attributes as in AHP in order to derive attribute weights and scores in spaces $\mathbb{M}$ and $\mathbb{A}$ respectively. The AHP′ method then consolidates, based on the AHP–MAUT relationship, these judgments into a preference structure defined in space $\mathbb{M}$.

Let x^t_{bj} be the best level for attribute j among all alternatives measured in the original scales in space $\mathbb{T}$. To simplify, assume all V^m_js are ratio scales in that their permissible transformations are constructed by multiplying V^m_j with a scalar for each attribute j. To derive the weights with respect to the MAUT scales, w^m_js, one can simply make pairwise comparisons between the best attribute levels among attributes and then apply the eigenvector approach to derive the normalized weights. For example, in making the ratio judgments between x^t_{b1} and x^t_{b2} between attributes 1 and 2, the decision maker is specifying the ratio between $w^m_1 V^m_1(x^t_{b1})$ and $w^m_2 V^m_2(x^t_{b2})$. Because all $V^m_j(x^t_{bj})$s are, by convention, equal to unity, the resulting ratio becomes w^m_1/w^m_2. All w^m_js can thus be uniquely determined from such pairwise comparison matrix among attributes through the eigenvector approach.

Similarly, in making pairwise comparisons between alternatives with respect to each attribute j, the decision maker is specifying the ratio of the attribute values. For example, comparing alternatives 1 and 2 with respect to attribute j implies that the ratio between the two is

$$\frac{w^m_j V^m_j(x^t_{1j})}{w^m_j V^m_j(x^t_{2j})},$$

which can be simplified to

$$\frac{V^m_j(x^t_{1j})}{V^m_j(x^t_{2j})}.$$

Through the eigenvector approach, the resulting scores, $V^a_j(x^t_{ij})$s, are, however, those measured in the AHP scales where the sum of $V^a_j(x^t_{ij})$s is equal to unity as shown in (6), not in the MAUT scales.

We have shown from (7) that the decision rule Z^{ma} by summing $w^m_j V^a_j(x^t_{ij})$s violates the condition in Theorem 1. It is necessary to transform either w^m_j to $w^{a'}_j$ or V^a_j to $V^{m'}_j$ in order to construct a preferentially consistent structure [retaining the relative worths among alternatives given by (9) or (5)], where $w^{a'}_j$ and $V^{m'}_j$ may or may not be equal to w^a_j and V^m_j respectively. The w^m_js are given by the decision maker as judgments and cannot therefore be transformed to w^a_js in (13) with V^m_js unknown to gain consistency. We, therefore, need to transform V^a_j to $V^{m'}_j$ to ensure a consistent structure. That transformation is given mathematically in Appendix 3, which allows w^m_js and V^a_js thus obtained to be consolidated into a coherent structure.

Therefore, instead of answering importance questions as in AHP, in AHP′ the decision maker first compares in pairs the relative worths between the best levels among attributes and then compares the relative worths between alternatives in pairs with respect to each attribute. The arithmetic computation of the relative worths among alternatives is based on (15) and (20). For example, compare two apartments, A and B, considering walking time (measured in min) to work and floor area (in ft^2). Assume A, being five minutes away from work, is closer to work than B (10 min) and B with 300 ft^2 has larger floor area than A with 200 ft^2. To derive the MAUT weights using the pairwise comparison approach, the decision maker would respond to the following question:

> "What is the ratio of worth between 5 min spent walking to work and 300 ft^2 of floor space?"

In making ratio judgments among alternatives with respect to attributes, say walking time to work, the decision maker would respond to the following question:

> "What is the ratio of worth between 5 min spent walking to work and 10 min?"

It is not clear, however, whether the decision maker is being asked to trade off unit for unit at that point or over the entire intervals in these questions. The clarification of this issue begs another paper. We concentrate here only on the scaling procedure. It is important to note that all the required conditions for additive multiattribute value functions as specified by [10] must be satisfied for such questions to be meaningful. The scaling methods of AHP′, AHP, and MAUT are summarized in Table 1. Note that AHP and AHP′ are different not only in elicitation questions, but also in the scales through which decision makers respond to these questions. In particular, AHP restricts ratio responses within a 1–9 scale, while AHP′ does not. An empirical comparison among AHP′, MAUT, and AHP has been done based on such transformations [14]. AHP′ did not perform as well as the other techniques, but it could be improved by asking questions easier to respond to.

DISCUSSION

As shown in Appendix 2, the rank reversal problem in AHP is purely a mathematical artifact, not something that results from a set of behavioral claims built into the mathematics. Thus the coincidence that these mathematics yield reversal and people exhibit reversal does not justify rank reversals, even for behavioral arguments. Appendix 2 also shows that the interdependence between alternatives and criteria is inherent in the computation of AHP. Saaty's claim in Axiom 3 that alternatives are independent of criteria is only for constructing a hierarchical structure [16]. There is a distinction between the computational and structural interdependences. Saaty's axiomatic foundation does not make this distinction or at least demonstrate the computational interdependence.

AHP′ is not a replacement for AHP, but a variant of it. It can also be viewed as a supplement to Saaty's axiomatic foundation in that the entries in pairwise comparison matrices are given explicitly as preference judgments. More specifically, the definition of the scale $P_c(A_i, A_j)$ not defined by Saaty in [16] is given in the present formulation. The scale is, in general, the ratio of two weighted scores in a certain attribute space $\mathbb{S}$, i.e.

$$\frac{w_i^s V_i^s(x_{ki}^l)}{w_j^s V_j^s(x_{lj}^l)},$$

where w_i^s and w_j^s are the coefficients (or weights or priorities) implying the trade-offs between attribute scores $V_i^s(x_{ki}^l)$ and $V_j^s(x_{lj}^l)$ in relation

Table 1. The interpretation of MAUT, AHP, and AHP′ based on the framework

Methods	MAUT	AHP	AHP′
Attribute spaces	$\mathbb{M}$	$\mathbb{A}$	$\mathbb{M}'$
Decision rules	$Z^m(x_{ij}^l) = \sum_{j=1}^{n} w_j^m V_j^m(x_{ij}^l)$	$Z^a(x_{ij}^l) = \sum_{j=1}^{n} w_j^a V_j^a(x_{ij}^l)$	$Z^{m'}(x_{ij}^l) = \sum_{j=1}^{n} w_j^m V_j^{m'}(x_{ij}^l)$
Coefficient methods of derivation	w_j^m tradeoff judgments among attributes	w_j^a paired comparisons among attributes (with 1–9 scale)	w_j^m paired comparisons among best attrib. levels (without 1–9 scale)
Unidimensional transformation functions	V_j^m	$V_j^a \equiv T_j \bigcirc V_j^m$	$V_j^{m'} \equiv T'_j \bigcirc V_j^a$
Methods of derivation	equivalence judgments within attributes	paired comparisons among attribute levels (with 1–9 scale)	paired comparisons among attribute levels (without 1–9 scale) then scaling by (20)

to the overall measurement for alternatives k and l in that space. We use the MAUT scales as an example to pinpoint the relationship between MAUT and AHP.

The rank reversal problem can be caused either by applying inappropriate attribute weights to attribute scores measured in a space, e.g. (7), or by addition or deletion of alternatives from the original set when AHP is applied [2]. The solution for the first type of rank reversal is to apply the appropriate weights to the attribute scores in a decision rule, both being defined with respect to the scales in the same space. The reason for the second type of rank reversal in AHP is that the addition or deletion of alternatives will change the scaling or normalization factor of (6). Thus, the resulting attribute scores are no longer measured by the same unidimensional transformation functions in one space, say $\mathbb{A}$, but by different functions in a different space $\mathbb{A}'$. The criteria priorities must be changed according to (13). Therefore, the rank reversal problem in AHP and the meanings of criteria priorities have a dual property in that retaining the relative worths among alternatives prescribes computation of priorities in a particular way, such as that in (13). It can also be generalized from the arguments justified that certain scaling methods for attributes prescribe the definition of weights, if the condition of mutually permissible transformation in Theorem 1 holds. AHP [with the weights and scores as defined in (13) and (6)], AHP′, MAUT, and the B–G Modified AHP are a set of different scaling methods resulting in such transformations.

CONCLUSIONS

In this paper, we provide a common language to compare multiattribute decision making techniques and use that language to interpret AHP based on preferences. A theorem of the condition under which two additive, multiattribute decision rules are mutually permissibly transformable is proved. The theorem implies the dual property between the rank reversal problem and the meaning of weights in AHP. The problem can be avoided if both the weights and attribute scores in a decision rule are derived from the scales in a common attribute space. Using MAUT and AHP as an example, the relationship between the two is given explicitly also based on that theorem. A new scaling technique, AHP′, is designed to incorporate both MAUT and AHP into a common logic in which the meaning of the entries in pairwise comparison matrices of alternatives is given as ratios of preferences.

APPENDIX 1

Conditions for Mutually Permissible Transformation

Theorem 1: *Let* $Z^u(x^t_{ij})$ *and* $Z^r(x^t_{ij})$ *be two decision rules as defined by* (1) *and* (2) *in spaces* $\mathbb{U}$ *and* $\mathbb{R}$ *composed of the same set of attributes respectively.* Z^u *and* Z^r *are mutually permissibly transformable if and only if* $w^u_j = k(\partial Z^r/\partial V^u_j)$, *for all* $j = 1, 2, 3, \ldots, n$, *where* $k > 0$ *is a constant of proportionality.*

Proof: For the sufficiency part, if Z^u and Z^r are ratio scales, then $Z^u = \alpha Z^r, \alpha > 0$. Taking partial derivatives on both sides with respect to each V^u_j respectively, we have

$$w^u_j = \frac{\partial \alpha Z^r}{\partial V^u_j} = \alpha \frac{\partial Z^r}{\partial V^u_j} = k \frac{\partial Z^r}{\partial V^u_j}, \quad j = 1, 2, 3, \ldots, n,$$

where $k = \alpha > 0$. If Z^u and Z^r are interval scales, then $Z^u = \alpha Z^r + \beta, \alpha > 0$. Taking partial derivatives on both sides with respect to each V^u_j respectively, we have

$$w^u_j = \frac{\partial \alpha Z^r}{\partial V^u_j} = \alpha \frac{\partial Z^r}{\partial V^u_j} = k \frac{\partial Z^r}{\partial V^u_j}, \quad j = 1, 2, 3, \ldots, n,$$

where $k = \alpha > 0$, which is desired.

For the necessity part, let $w^u_j = k(\partial Z^r/\partial V^u_j), j = 1, 2, 3, \ldots, n$. Since w^u_j is a constant and thus continuous with respect to V^u_j, the general antiderivative of w^u_j with respect to V^u_j is $w^u_j V^u_j(x^t_{ij}) + c$, where c is a constant. We thus have $w^u_j V^u_j(x^t_{ij}) = kZ^r(x^t_{ij}) - c, i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$. Summing the weighted attribute values (items on the left side) in space $\mathbb{U}$ across all attributes j for some alternative i, we have

$$\sum_{j=1}^{n} w^u_j V^u_j(x^t_{ij}) = Z^u(x^t_{ij}) = n[kZ^r(x^t_{ij}) - c]$$

$$= nkZ^r(x^t_{ij}) - nc = \alpha Z^r(x^t_{ij}) + \beta(\alpha = nk; \beta = -nc);$$

$\alpha > 0$ because $k > 0$, which completes the proof. Note that if $c = 0$, Z^u and Z^r are a ratio scale; otherwise, they are an interval scale. It is obvious that Theorem 1 holds as long as the attribute scores in the arguments of Z^u and Z^r are defined in the attribute spaces with the same dimensionality, i.e. x_{ij}^t can be replaced by v_{ik}^s in any attribute space $\mathbb{S}$ with a dimensionality different from the original one, i.e. $j \neq k$.

APPENDIX 2

The Rank Reversal Problem and its Solution

Mathematically, let

$$Z^{ma}(x_{ij}^t) = \sum_{j=1}^{n} w_j^m V_j^a(x_{ij}^t)$$

be a decision rule in the AHP space $\mathbb{A}$, representing the multiplication of the MAUT weights with the AHP priorities of alternatives. Substituting (6) into this decision rule, we have

$$\begin{aligned} Z^{ma}(x_{ij}^t) &= \sum_{j=1}^{n} w_j^m V_j^a(x_{ij}^t) \\ &= \sum_{j=1}^{n} w_j^m T_j \bigcirc V_j^m(x_{ij}^t) \\ &= \sum_{j=1}^{n} k_j w_j^m V_j^m(x_{ij}^t), \end{aligned} \tag{7}$$

where

$$k_j = \left[\sum_{i=1}^{m} V_j^m(x_{ij}^t) \right]^{-1}.$$

Taking the partial derivative on Z^{ma} with respect to each $V_j^m(x_{ij}^t)$, we have

$$\frac{\partial Z^{ma}}{\partial V_j^m} = k_j w_j^m. \tag{8}$$

Because k_j varies in relation to j, the condition in Theorem 1 does not hold. We can, therefore, conclude that Z^m and Z^{ma} are not mutually permissibly transformable. The ranking or relative worths among alternatives derived from Z^m and from Z^{ma} may not be the same. Therefore, the weights w_j^ms defined in relation to the MAUT scales must be modified in (7) to retain the relative worths of alternatives. How these weights should be modified in order to preserve relative worths among alternatives is the question we next address.

In AHP, the multidimensional additive decision rule for three-level hierarchic structures, i.e. the composition rule, is similar to that in (5) as shown in (9)

$$Z^a(x_{ij}^t) = \sum_{j=1}^{n} w_j^a V_j^a(x_{ij}^t), \quad \text{for } i = 1, 2, \ldots, m, \tag{9}$$

where

Z^a = the multidimensional additive decision rule for AHP in space $\mathbb{A}$,
w_j^a = the attribute weight in relation to the AHP scales for attribute j, and
V_j^a = the two-step unidimensional transformation function sending the measurement for attribute j from space $\mathbb{T}$ to space $\mathbb{A}$ through $\mathbb{M}$.

Substituting (6) into (9) to derive the meaning of w_j^a in terms of w_j^m and V_j^m, we have

$$Z^a(x_{ij}^t) = \sum_{j=1}^{n} \frac{w_j^a}{\sum_{k=1}^{m} V_j^m(x_{kj}^t)} V_j^m(x_{ij}^t), \quad \text{for } i = 1, 2, \ldots, m. \tag{10}$$

In order for (10) to be preferentially consistent with (5), i.e. to retain the same relative worths among alternatives or for (10) and (5) to be mutually permissibly transformable, the trade-off ratios between the attributes with respect to the values in the MAUT scales must remain the same. That is, if Z^a and Z^m are mutually permissibly transformable, by Theorem 1

$$\begin{aligned} w_j^m &= k \frac{\partial Z^a(x_{ij}^t)}{\partial V_j^m(x_{ij}^t)} \\ &= k \frac{w_j^a}{\sum_{k=1}^{m} V_j^m(x_{kj}^t)}, \quad \text{for } j = 1, 2, \ldots, n, \end{aligned} \tag{11}$$

where $k > 0$ is a constant. From (11)

$$\begin{aligned} w_j^a &= k^{-1} w_j^m \sum_{k=1}^{m} V_j^m(x_{kj}^t) \\ &= k^{-1} \sum_{k=1}^{m} w_j^m V_j^m(x_{kj}^t), \quad \text{for } j = 1, 2, \ldots, n. \end{aligned} \tag{12}$$

Because AHP requires that all weights sum to unity,

$$w_j^a = \frac{\sum_{i=1}^{m} w_j^m V_j^m(x_{ij}^t)}{\sum_{l=1}^{n}\sum_{k=1}^{m} w_l^m V_l^m(x_{kl}^t)}, \quad \text{for } j = 1, 2, \ldots, n. \quad (13)$$

With (6) and (13), the solution to the rank reversal problem caused by inconsistent weights and scores implies the relationship between MAUT and AHP. Conversely, w_j^m and V_j^m can also be represented in terms of w_j^a and V_j^a [12].

APPENDIX 3

Scaling Transformation from V_j^a to $V_j^{m'}$ in AHP'

Let T_j' be the transformation function. We have

$$V_j^{m'}(x_{ij}^t) = T_j' \bigcirc V_j^a(x_{ij}^t), \quad \text{for } i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n, \quad (14)$$

where $V_j^{m'}$ is a unidimensional transformation function sending attribute j from space $\mathbb{T}$ to space $\mathbb{M}'$. Replace V_j^a with $V_j^{m'}$ in (7). We then have

$$Z^{m'}(x_{ij}^t) = \sum_{j=1}^{n} w_j^m V_j^{m'}(x_{ij}^t) = \sum_{j=1}^{n} w_j^m T_j' \bigcirc V_j^a(x_{ij}^t) \quad \text{for } i = 1, 2, \ldots, m, \quad (15)$$

where $Z^{m'}$ is the new decision rule defined in space $\mathbb{M}'$. In order for Z^m and $Z^{m'}$ in spaces $\mathbb{M}$ and $\mathbb{M}'$ respectively to be mutually permissibly transformable, the condition in Theorem 1 requires that

$$w_j^m = k\frac{\partial Z^{m'}}{\partial V_j^m} \quad \text{for } j = 1, 2, \ldots, n. \quad (16)$$

The sufficient condition ensuring that (16) holds is that $V_j^{m'}$ is a positive linear transformation of V_j^m with a common scalar, implying that either T_j' is positively linear with respect to V_j^a, which is in turn positively linear with respect to V_j^m or both are negative linear transformations. Negative, unidimensional linear transformations are not included in the set of permissible transformations according to Theorem 1 because any such transformation may result in $k < 0$. Both T_j' and V_j^a must, therefore, be positive linear transformations with respect to V_j^a and V_j^m respectively. For simplicity, we do not include the constant terms in the linear transformations because we have ratio scales. Therefore, we have

$$T_j'(x_{ij}^t) = \alpha_{tj} V_j^a(x_{ij}^t) \quad \text{for } i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n \quad (17)$$

and

$$V_j^a(x_{ij}^t) = \alpha_{vj} V_j^m(x_{ij}^t), \quad \text{for } i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n, \quad (18)$$

where α_{tj} and α_{vj} are both greater than zero. Substituting (18) into (17), the composite function $T_j' \bigcirc V_j^a$ in (15) becomes

$$V_j^{m'}(x_{ij}^t) = T_j' \bigcirc V_j^a(x_{ij}^t) = \alpha_{tj}\alpha_{vj} V_j^m(x_{ij}^t), \quad \text{for } i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n. \quad (19)$$

Substituting (19) into (15), taking partial derivatives with respect to each V_j^m, and comparing the result with (16), the product $\alpha_{tj}\alpha_{vj}$ is equal to the constant k for all js, implying that the trade-offs among attribute values in terms of V_j^m remain the same between Z^m and $Z^{m'}$. α_{tj} and α_{vj} should, therefore, be so determined that their product is a constant for all js. For example, α_{vj} is the coefficient in (18) transforming V_j^m to V_j^a. If α_{vj} is defined as in (6), i.e.

$$\alpha_{vj} = \left[\sum_{k=1}^{m} V_j^m(x_{kj}^t)\right]^{-1},$$

then α_{tj} must be a multiplier of the reciprocal of α_{vj} so that the product of the two is a constant. Since all V_j^ms are unknown, α_{vj} can be any arbitrary value and thus α_{tj} cannot be uniquely determined. By convention, the maximum of $V_j^m(x_{ij}^t)$ is, however, set to unity. With this further information, we can uniquely determine α_{vj} and α_{tj}.

Let $V_j^m(x_{bj}^t)$, the value for the best level of attribute j, be equal to unity. Because V_j^a is, according to (6), a monotonic increasing transformation function of V_j^m, $V_j^a(x_{bj}^t) = \alpha_{vj} V_j^m(x_{bj}^t)$ must also be the maximum among all $V_j^a(x_{ij}^t)$s. Therefore, let $\alpha_{vj} = V_j^a(x_{bj}^t)$. We have $\alpha_{tj} = k[V_j^a(x_{bj}^t)]^{-1}$ because $\alpha_{tj}\alpha_{vj} = k$. Substituting α_{vj} and α_{tj} thus obtained into (18) and (19), we have

$$V_j^{m'}(x_{ij}^t) = k\frac{V_j^a(x_{ij}^t)}{V_j^a(x_{bj}^t)}, \quad \text{for } i = 1, 2, \ldots, m. \quad (20)$$

The transformed scores, $V_j^{m'}(x_{ij}^l)$, can then be multiplied by w_j^ms according to (15) to compute the relative worths among alternatives consistent with those derived from (5) and (9). Note that when $k = 1$, i.e. $\alpha_{ij}\alpha_{vj} = 1$, according to (19) $V_j^{m'}(x_{ij}^l) = V_j^m(x_{ij}^l)$. To derive from normalized scores for alternatives with respect to each criterion in AHP the MAUT values for the corresponding attribute, we can simply divide these normalized scores by the maximum for each attribute or criterion.

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