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# On transition rules of complex structures in one-dimensional cellular automata: Some implicaltions for urban change

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**Abstract.** Based on the assumptions that cities are semi-lattices and that their spatial configuration are complex structure, I use one-dimensional cellular automaton representing a hypothetical, linear city as an analytic tool to investigate possible transition rules fulfilling these requirements, and base on that metaphor to draw some implications for urban change. For two-value ($k = 2$), one-neighbor ($r = 1$) one-dimensional cellular automata, the stochastic transition rules thus found imply that determinism at one level can give rise to stochasticity at another level, and that the seemingly stochastic processes of urban change might indeed be governed by a few deterministic transition rules.

**JEL classification:** R00, R10, R14

## 1. Introduction

The paper investigates explanations of the local-global interaction of urban spatial systems, with a focus on one-dimensional cellular automata representing hypothetical, linear cities as an analytical tool for urban change. It is grounded on the hypothesis that the global characteristics of complex spatial systems, such as cities, emerge from the local interaction among the elements consisting of these systems, such as individual agents in an economy (e.g., Holland 1995, p. 1, 41–42; Krugman 1996, p. 21; Simon 1998, p. 33–34). Though it is arguably true that using one-dimensional cellular automata may render the analytical frames and results unrealistic or too simplistic, analyzing ‘long, narrow’ cities has been a long tradition in urban economic theory (Krugman 1996, p. 22). I follow that tradition here by focusing on interaction

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among local agents, hoping that this new approach would yield useful insights into our understanding of how urban systems evolve. The physical environment of a city is the outcome of interacting local development decisions. Models on urban evolution based on the top down approach but ignoring the local interaction imply that planning can be carried out in a similar way, e.g., vertical, centralized organizations, comprehensive process, and generic policies for land development. With such a conception distinct from the fundamental characteristics of spatial evolution, urban change is thus difficult to be tamed in reality by traditional planning techniques because of the complexity of relatedness among spatial decisions (Batty 1995). Insight into effective planning techniques may be gained through the bottom up approach to urban change, i.e., understanding how the local interaction among the development decisions affects the overall trend in the urban physical change. Therefore, I tend to explore into such possibility by focusing on the local-global interaction at the most fundamental, abstract level. I thus set aside the substantive interpretation of or elaboration on the abstract structure as future work. Obviously, the price of abstractness is the deviation from realism, but if interpreted appropriately, we can still gain useful insight from the abstract construct. A similar example following this line of research is conducted by Hillier (1996), but he did not focus on the formalization of the conceptual models of spatial organization, which I set as my long-term objective. Thus the paper serves as a starting point toward that end. I have, however, conducted concrete computer experiments searching for the order out of chaotic evolution of complex spatial systems, and reported my findings elsewhere (Lai 1999). Different from most current two-dimensional cellular automata spatial simulations, the present paper explores deductively into the mechanisms underlying the emergence of complex structures using one-dimensional cellular automata representing hypothetical, linear cities.

There is a shift recently in paradigm on urban spatial modeling approaches from the top down approach that views the aggregate pattern in urban change as equilibrium seeking to the bottom up approach that considers the seemingly stable pattern as emerging from the dynamics and interaction among local actions (Batty 1996). This perspective of understanding urban change is in part influenced by a new orientation in science that concerns the behavior of complex systems: complexity.

Most planning behavior takes place in complex settings. The elements in these settings interact with each other not only forming a complex network of information flow, but also resulting in uncertainty or incomplete information faced by planners. Understanding the nature of the complex system is helpful in developing a prescriptive theory of planning to aid planners to cope with uncertainty. Recent development in such understanding leads to a set of related new fields, including artificial life (e.g., Emmeche 1994) and complexity theory (e.g., Gell-Mann 1994).

The central idea of artificial life is that simple rules result in complex behavior of a system. Artificial life researchers conduct their experiments on computers by assuming that life can emerge from very simple rules and creating their own universes or games of life on computers. The validity of these computer experiments is being debated, but they may be efficient tools for discovering the simple rules of nature, which can possibly be replicated in the real experiments. The ultimate objective of the artificial life research is to search for the plausible laws of nature for further mathematical proof.

The computer experiment approach used in the artificial life research is quite promising for developing ill-understood theories.

An even more powerful, ambitious theory encompassing a wider range of phenomena is complexity theory (Gell-Mann 1994). It is an attempt of a crude integration of the current state of knowledge scattered across various disciplines, including theoretical physics, psychology, computer science, political science, and economics. The objective of the theory is to discover complex adaptive systems in nature and explain their behavior. It is distinct from the traditional systems approach in that complexity theory can, to some extent, be thought of as a bottom up approach, whereas the traditional systems approach is a top down one. Many urban dynamical and spatial models are based on the latter approach (e.g., Forrester's urban system dynamics model 1973). Complexity theory seeks the fundamental laws of how a system adapts to interference, exogenous or endogenous. Information is an important measure of describing the system's behavior. The systems approach tends to divide the whole system into functional components. The relationship among these components is subject to rigorous tests to verify the model. No fundamental laws are required in building that model. Complexity theory provides a powerful perspective of interpreting observed social and natural phenomena, but its development is still in an early stage.

There are other earlier works on complexity, most notably Simon (1998) and Alexander (1965), which are more pertinent to our discussion of land use change in cities. Simon considered the complex structures emerging in nature as "nearly decomposable systems", meaning the elements in these systems are organized in an almost hierarchical form of structures. But unlike pure hierarchies where the elements are related only from the top down, the elements at the same level in these complex structures are interrelated among each other. Such structures have the advantage of growing fast, thus resulting in higher probabilities of being existent. In an attempt to search for the organizational principles of how natural cities grow, Alexander proposed a generic structural principle called "semi-lattice" similar to Simon's idea about complexity for spatial organization. In contrast to a tree, in this principle the relationship among elements of a system is similar to that in what Simon called nearly decomposable system in that if any two elements have common subsets of the system, then these subsets also belong to the system. A tree structure where the relationship among elements is based on which belongs to which is thus by definition a semi-lattice structure. The relationship underlying most semi-lattice structure is much more complex and richer than that in a simple tree structure. Alexander (1965) argued that cities grow following the semi-lattice principle, and planners should recognize such a principle and provide appropriate spatial structures accordingly.

Cellular automata, as a metaphor of urban change in linear cities used here, are a branch of complexity theory. They are the simplest models of investigating the local-global interaction in complex systems. The research on cellular automata was pioneered by von Neumann (1966) and thoroughly investigated by Wolfram (1994c, 2002). The behavior of cellular automata has not been fully understood, but researchers have gained enough experience in order to formalize its dynamics (e.g., Urias et al. 1996). There are at least two approaches to understand the cellular automata behavior, computer experiments (e.g., Wuensche and Lesser 1992 and Wolfram 2002) and mathematical deduction (e.g., Urias et al. 1996 and Wolfram 1994a, b). Distinct from computer simulations using two-dimensional cellular, I intend to explore into urban change metaphorically based on the mathematical deduction approach. General structural transition rules of how spatial agents interact that characterize complex spatial systems can be derived from the deduction as will be depicted shortly. A promising research agendum would be to incorporate a set of hypothetical structural considerations, such as Alexander's semi-lattice structure, into the cellular automata model by designing or discovering evolution rules according to the logic of that structure. Through deductive reasoning based on automata theory and mathematical logic (e.g., Hopcroft and Ullman 1979 and Mendelson 1987), we could then look into the trajectories of the system to generalize how each of such rules affects the evolutionary outcomes. Finally, we could evaluate how well each rule thus selected in the cellular automata model maps the real urban dynamics on the empirical urban growth indices, such as population and morphology of urban boundaries.

Therefore, the main construct underlying our exploration is cellular automata, in particular those of one dimension, that have been used to investigate evolution of complex systems (e.g., Wolfram 1994c, 2002). They are recently being applied in the explanation of or experimentation on urban development (e.g., Couclelis 1985 and Cecchini 1996). Instead of modeling the reality of urban change based on cellular automata as most such work seems to imply, I use one-dimensional cellular automata here as an analytic tool representing hypothetical, linear cities to examine how complex spatial structures emerge. That is, I view cities as complex spatial system reminiscent of those emerging from cellular automata and search for possible rules that govern their evolution.

Section 2 introduces briefly the one-dimensional cellular automata model and distinguishes between a tree and semi-lattice rules based on the state transition graphs. Section 3 classifies the 256 transition rules for the model where $k$ (number of cell values or states) and $r$ (number of interacting neighbors) equal to 2 and 1 respectively in terms of rule types and Wolfram's four-class categories (1994b). In particular, a characteristic transition rule is derived from the eight rules that are classified as semi-lattices and exhibit complex structures. Section 4 provides implications for urban change based on the characteristic transition rule found in Sect. 3 and discusses some spatial issues and possible future extension of the model.

## 2. The model

Consider a linear city of a finite set of spatial agents or blocks represented by a one-dimensional cellular automaton. Assume each block can be of one of two types of land use, retail with a value of one or residential of zero. There can be many ways in which the linear city could evolve in space and time. Grounded on the assumption that cities are complex structures in the form of semi-lattices, the question is: Are there underlying mechanisms of the interaction of these blocks in the linear city that give rise to semi-lattice structures. There are at least three ways of defining the semi-lattice form of one-dimensional cellular automata models: 1) direct specification of transition rules (i.e., how blocks affect each other in land use); 2) pattern recognition of configurations through time in the time-space plots; and 3) pattern recognition of configurations of space at each time step. The common framework on which the three definitions are based to identify the semi-lattice form of cellular automata models is automata theory and formal languages. Since there is no easy way to define the semi-lattice form through pattern recognition for the one-dimensional cellular automata, I focus here on the first approach to defining semi-lattices. The evolution of a one-dimensional cellular automaton can be viewed as a set of languages generated by a finite automaton. Following Wolfram (1994a), let the value (land use type) of site (block) $i$ at time step $t$ be denoted as $a_i^{(t)}$ and be a symbol selected from the alphabet

$$S = \{0, 1, \ldots, k-1\} \tag{1}$$

All possible sequences of these symbols form the set $\Sigma$ of cellular automaton configurations $A^{(t)}$. At each time step each site value is updated according to the values of a neighborhood of $2r+1$ sites around it by a local (or transition) rule (determining how the blocks interact)

$$\phi : S^{2r+1} \to S \tag{2}$$

of the form

$$a_i^{(t)} = \phi[a_{i-r}^{(t-1)}, a_{i-r+1}^{(t-1)}, \ldots, a_{i+r}^{(t-1)}]. \tag{3}$$

This transition rule leads to a global mapping

$$\Phi\colon \Sigma \to \Sigma. \tag{4}$$

on the complete cellular automaton configurations. Let $\Omega^{(t)}$ denote the set of configurations generated after $t$ iterated applications of $\Phi$ on $\Sigma$, i.e.,

$$\Omega^{(t)} = \Phi^t \Sigma. \tag{5}$$

There is an economic way of representing all possible configurations generated by $\Phi$ over $t$ time steps on $\Sigma$ based on a so called state transition graph for the non-deterministic finite automaton (NDFA) corresponding to $\Phi$ (c.f., Appendix A). Take rule 76 as defined by Wolfram (1994a) (c.f., Appendix B). Figure 1 shows the corresponding NDFA of the rule.

Each node represents a state of the automaton. Each arc with a symbol (0 or 1) is the mapping from a subset of three neighbors at the previous time step to a symbol for the central site of the subset at current time step. For example, the arc from node 11 to 10 with the symbol of 1 represents the mapping from 110 to 1. The finite automaton is non-deterministic because there are same symbols emanating from a particular set of nodes. This means that transformations from these nodes cannot be determined definitely. For example, node 00 has two arcs labeled 0 emanating from it to node 01 and itself. Thus the finite automaton transition graph for rule 76 in Fig. 1 is non-deterministic. For each non-deterministic finite automaton, there exists at least a corresponding deterministic finite automaton (DFA) generating the same language based on subset construction (Hopcroft and Ullman 1979). Take rule 76 again. The state transition graph for one such corresponding deterministic finite automaton is shown in Fig. 2. Since there can be more

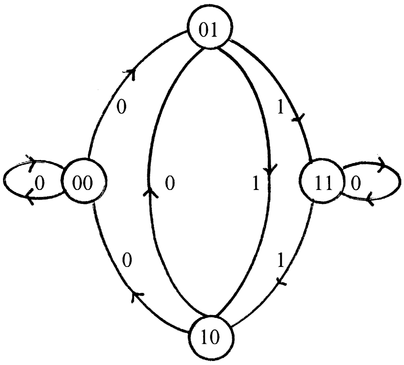


**Fig. 1.** The NDFA corresponding to rule 76

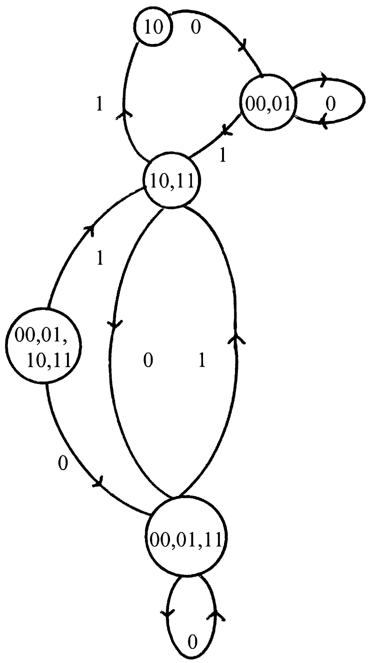


**Fig. 2.** A DFA corresponding to the NDFA for rule 76

than one DFA corresponding to an NDFA, for my classification purpose I developed an algorithm to derive the minimal DFA for each transition rule, that is uniquely associated with the NDFA for that rule (c.f., Appendix A).

As depicted earlier, Alexander's (1965) notion of semi-lattice is the original idea concerning spatial overlaps of categories, whereas I use one-dimensional cellular automata here as the simplest discrete dynamic system mimicking evolution of the linear urban system. Even though the relationship between the semi-lattice in the spatial context of urban systems and those in the cellular automata rules is difficult to pin down analytically, they are at least topologically equivalent. Alexander's original idea of spatial overlaps is based on set theory and topology, which is also the theoretical foundation of my approach to the typology of the one-dimensional cellular automata rules. There might be links between the two, but they fall outside the scope of the present paper. Since both Alexander's and my expositions are based on the same theoretical foundation, at a higher level, the notion of semi-lattice can be used here to examine its relation to urban change. Based on Alexander's (1965) distinction between trees and semi-lattices, we can define two types of transition rules according to the concept of deterministic finite automata.

**Definition 1. The tree rule.** *A transition rule is called a tree rule if and only if, when one node is a subset of another in the state transition graph corresponding to the minimal DFA obtained from the NDFA, there exists at least one arc connecting the two nodes.*

**Definition 2. The semi-lattice rule.** *A transition rule is called a semi-lattice rule if and only if, when two nodes in the state transition graph corresponding to the minimal DFA obtained from the NDFA have common elements (or subsets of the nodes in the NDFA), there exists at least one arc connecting the nodes. If a transition rule includes both the tree and semi-lattice cases, it is classified as a semi-lattice rule.*

According to these definitions, the DFA for rule 76 as shown in Fig. 2 belongs to the semi-lattice structure. Appendix A depicts how the DFA is constructed. In order to determine the DFAs uniquely for the purpose of rule classification, I graphed the minimal ones with the smallest numbers of arcs and nodes. The minimal DFAs thus obtained should be unique (c.f., Appendix A). It turns out that rule 76 indeed belongs to semi-lattices. By searching out all such rules in a given one-dimensional cellular automaton, we can find their general characteristics of its evolution and analyze their behavior as will be shown on the following sections.

## 3. Simulation design and general observations

According to the patterns of evolution in the space-time configurations, one-dimensional cellular automata can be classified into four categories (Wolfram 1994b).[1] The four general classes are:

*Class 1*: A fixed, homogeneous state is eventually reached;
*Class 2*: A pattern consisting of separated periodic regions is produced;
*Class 3*: A chaotic, aperiodic pattern is produced; and
*Class 4*: Complex, localized structure are generated.

Class 4 structures are of particular interest because theoretically they are capable of universal computation and reminiscent of Game of Life (Wolfram 1994 and Berlekamp et al. 1985). I argue that if the spatial evolution of cities can be viewed as governed by the interaction rules yet to be found similar to those in cellular automata, the changing spatial configurations of cities suggest that the Class 4 structures would most likely, at least in the simplest case of a linear city, represent such evolution. Buildings are being constructed and torn down; factories and stores being opened and closed; houses being built and abandoned; population concentrations prospering and declining; and commercial and residential areas moving from one place to another. Behind all these dynamics might lie the fundamental rules according to which individual agents interact spatially. With the two structural assumptions on the spatial evolution of cities depicted earlier (i.e., semi-lattice rules and complex struc-

[1] Though Wolfram claimed that "approximately" there are no Class 4 structures in one-dimensional cellular automaton with $k = 2$ and $r = 1$ (1994b), the observation seems inconclusive because it is only an "approximate" estimation. In the paper, I summarized all the 256 rules in terms of the four classes by running the program provided by Wuensche and Lesser (1992). The number of the rules yielding the Class 4 structure was indeed small (only eight out of 256), which, I think, is consistent with Wolfram's findings. Even though these structures are much simpler than the Class 4 structures found in other one-dimensional cellular automata with greater $k$s and $r$s, they cannot be apparently classified into any of the rest of the three classes. The simplicity of the structure might be caused by the simplicity of the rules with $k = 2$ and $r = 1$, but we cannot thus conclude that there are no Class 4 structures in this model.

**Table 1.** Classification of transition rules for one-dimensional cellular automata with $k = 2$ and $r = 1$ by types of rules and classes of structures

| | Semi-lattice rules | Tree rules | Others | Total |
|---|---|---|---|---|
| Class 1 structures | 58 | 24 | 2 | 84 |
| Class 2 structures | 87 | 32 | 4 | 123 |
| Class 3 structures | 25 | 4 | 12 | 41 |
| Class 4 structures | 8 | 0 | 0 | 8 |
| Total | 178 | 60 | 18 | 256 |

**Table 2.** The eight transition rules that are semi-lattices and result in class 4 structures

| Rule number | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 |
|---|---|---|---|---|---|---|---|---|
| 9 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 41 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 65 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 97 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| 107 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 |
| 111 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 121 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| 125 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |

tures) and imposed on the one-dimensional cellular automaton, we can search out these transition rules conforming to the assumptions from which insight into the origin of urban spatial evolution might be gained.

Using the one-dimensional cellular automaton with $k = 2$ (two uses of land) and $r = 1$ (number of blocks affecting each other), I first grouped the 256 rules into the semi-lattice, tree, and the remaining rules as defined earlier. I then further classified each group of rules according to Wolfram's four-class categories. The following table shows the result. The detailed classification of the transition rules is given in Appendix B.

It can be found from Table 1 that almost all transition rules are semi-lattices or trees. Among these rules, 70.0% of the total are semi-lattices. The number and percentage of transition rules that result in the Class 4 structure (8 rules and 4.5%) are relatively greater for the semi-lattice rules than those for the tree rules (0 rules and 0.0%). The proportion of the transition rules yielding the Class 4 structure is significantly low among all transition rules (3.1%). Note that the transition rules resulting in the Calss 4 structure are all semi-lattices. A closer examination of the transition rules that are semi-lattices and yield the Class 4 structure can be used to generalize the characteristics of such rules. Table 2 shows the eight rules within this category.

The general characteristics of the eight rules can be summarized as below (see Table 3):

(1) If the central cell (block) has the same value (type of land uses) as that of the two neighbors at time $t$ (000 and 111), its value will change at time $t + 1$.
(2) If the central cell has the value of zero at time $t$ with its two neighbors having different values (001 and 100), there is a probability of 3/4 that the

**Table 3.** The characteristic transition rule for semi-lattices with complex structures

| State of surrounding | State of cell under consideration | |
|---|---|---|
| | Live cell (1) | Dead cell (0) |
| Two live cells | Dead | 1/4 Chances of being dead |
| One live and one dead cells | 3/4 Chances of being live | 3/4 Chances of being dead |
| Two dead cells | 1/4 Chances of being live | Live |

value of that cell will remain the same, while 1/4 that the value will change at time $t+1$.

(3) If the central cell has the value of zero at time $t$ with its two neighbors having the values of one (101), there is a probability of 3/4 that the value of that cell will change, while 1/4 that the value will remain the same at time $t+1$.

(4) If the central cell has the value of one at time $t$ with its two neighbors having the values of zero (010), there is a probability of 3/4 that the value of that cell will change, while 1/4 that the value will remain the same at time $t+1$.

(5) If the central cell has the value of one at time $t$ with its two neighbors having different values (110 and 011), there is a probability of 1/4 that the value of that cell will change, while 3/4 that the value will remain the same at time $t+1$.

## 4. Implications and discussion

The characteristic transition rule found here may shed some promising light on understanding the origin of urban change. Firstly, viewing the values or states of cells as different land uses, we can observe how these land uses interact spatially. Consider, for example, retail and residential uses as live (whose value is one) and dead (whose value is zero) cells respectively. The characteristic transition rule shows that when there are only retail uses in a neighborhood with no residential uses, the central site will change from the retail to residential use, whereas when there are only residential uses in that neighborhood with no retail uses, the central site will change from residential to retail use. This is intuitively plausible because the residential uses form the market for the retail uses, and without the market, the retail uses cannot survive. Secondly, the characteristic transition rule is stochastic, implying that determinism at one level can give rise to stochasticity at another level. The seemingly probabilistic processes of urban evolution might indeed be governed by a few deterministic interaction rules.

The proportions of the 256 rules in terms of rule types and classes in Table 1 imply that a tree rule may lead to a structure different from a semi-lattice rule. The crux is, however, that the eight rules found resulting in the Class 4 structure are all semi-lattices. This means that the transition rules embedded in the complex structure are themselves semi-lattices. More realistically, urban systems may be viewed as two-dimensional cellular automata. It is very likely that these systems are also the Class 4 structures in the time-space plots because two-dimensional cellular automata also appear to exhibit the four classes of structures identical to those in one-dimensional cellular automata (Wolfram 1994b). The implication is that the deterministic transition rules that govern the evolution of urban systems are semi-lattices. Put differently, though Alexander's definition of semi-lattices is concerned with spatial overlaps in a city and there is no simple mapping between the semi-lattices in the spatial context of urban systems and those in automata theory, it is likely, as my findings seem to imply, that the two definitions might be closely related. The spatial overlaps in terms of semi-lattices might imply that the dynamic representation of these overlaps, e.g., the minimal DFAs in the one-dimension cellular automaton, is also a semi-lattice. That is, they are at least topologically equivalent.

It is too simple minded at present to argue that urban spatial evolution indeed follows the analysis suggested here. As put earlier, I use the one-dimensional cellular automata model only as a representation of a hypothetical, linear city to understand the origin of urban change, setting aside the substantive meanings of the dynamics for future work. I have, however, conducted computer experiments based on a two-dimensional cellular automaton where the transition rules could evolve and land developers could learn these rules over time (Lai 1999). Preliminary findings showed that the complex spatial system tended to self-organize itself toward a critical state (Bak and Chen 1991). The implication is that the complex structure in the two-dimensional cellular automaton and the self-organizing process might be closely related.

To render the findings useful in planning, the immediate future work is to determine whether urban spatial models built on the rules similar to the characteristic transition rule can map the real data of urban change. Based on the hypothesis that the dynamics of cities have common characteristics across all scales, we need first to search for a general, scale free principle governing the dynamics based on which to evaluate the alternative models using the real data, such as the growth of populations. One possible alternative of such principle is Stanley et al's (1996) work on scaling behavior in the growth of companies. Based on the data on growth rates of sales of all US manufacturing publicity traded companies, Stanley et al (1996) found the growth rates could be all scaled according to an exponential distribution function and collapsed across all scales of sales into a single distribution given the scaling functions of parameters in the exponential distribution. Stanley et al thus concluded that "these findings are reminiscent of the concept of universality found in statistical physics, where different systems can be characterized by the same fundamental laws, independent of 'microscopic' details". Extending Stanley et al's models on growth rates of companies, we can search for a similar principle that governs growth rates of cities based on the real data on populations for all scales of cities. The cellular automata models on urban change derived from the characteristic transition rule are then evaluated according to the principle. The most effective model should yield the prediction of the population growth across all scales of cities closest to the reality.

## 5. Conclusions

Grounded on the hypothesis that the spatial evolution of urban systems can be characterized by the local interaction of individual agents, I expect to gain progress ultimately in understanding the origin of urban change from the new, bottom up perspective that has arisen in recent years. In the present paper, I have found, at least for a hypothetical, linear city of one-dimensional cellular automata, that the complex structure is derived from a set of transition rules whose dynamic representations are semi-lattices, not trees. These deterministic transition rules can be further grouped into a stochastic transition rule that gives rise to the Class 4 structure. The implication is that the evolution of urban systems, when viewed as cellular automata, might be governed by a few deterministic transition rules which are semi-lattices in the dynamic representation. A few deterministic rules might indeed be embedded in the seemingly stochastic processes of urban evolution.

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## Appendix A
## The minimization algorithm of transforming NDFAs to DFAs

Since an NDFA for a rule corresponds more than one DFA for the same rule, it is necessary to find a unique representation of the DFA for that rule in order to determine whether the rule is a tree or semi-lattice. One way to accomplish this is to transform the NDFA to the unique DFA by the minimization algorithm given by Wolfram (1994a) and Hopcroft and Ullman (1979). Using rule 76 as an example, I propose here a three-step operational version of the algorithm based on which the 256 rules are classified.

*Step 1.* Initiate the binary relations from the NDFA

Denote nodes 00, 01, 10, and 11 by $u_0, u_1, u_2$ and $u_3$ respectively. The NDFA for rule 76 as shown in Fig. 1 can be represented by the following binary relations:

$$u_0 \to 0u_0,\ u_0 \to 0u_1,\ u_1 \to 1u_3,\ u_1 \to 1u_3,\ u_1 \to 1u_2,\ u_2 \to 0u_1,$$
$$u_2 \to 0u_0,\ u_3 \to 0u_3,\ \text{and}\ u_3 \to 1u_2, \tag{6}$$

where the left hand side in a relation is the current state (node), while the right hand side the next state, and the number is the output of the transition.

*Step 2.* Construct a DFA based on sequential subset construction

Let $\sigma$ be the set of all possible subsets of nodes $u_i$, for $i = 0, 1, 2,$ and 3. There exit totally $2^4$ or 16 possible subsets. Each subset is a candidate for the nodes in a corresponding DFA, and the starting node for the DFA is defined as the subset $\sigma^* = \{u_0, u_1, u_2, u_3\}$. Consider the set of relations in (6) and proceed from the starting node. Given the output value, i.e., the number on the right hand side, as zero, the elements in the starting node, i.e., $u_0, u_1, u_2,$ and $u_3$ are transformed, if any, into the elements in the subset $\{u_0, u_1, u_3\}$, which is represented by an extended relation $\{u_0, u_1, u_2, u_3\} \to 0\{u_0, u_1, u_3\}$. Similarly, given the output value as unity, we can obtain another extended relation $\{u_0, u_1, u_2, u_3\} \to 1\{u_2, u_3\}$. Apply the same logic to the resulting subsets on the right hand side in the above relations sequentially until no new subsets are enumerated, and we can construct the extended relations in the DFA as shown below. Note that not all the 16 subsets are included in the set of extended relations.

$$\{u_0, u_1, u_2, u_3\} \to 0\{u_0, u_1, u_3\},\ \{u_0, u_1, u_2, u_3\} \to 1\{u_2, u_3\},$$
$$\{u_0, u_1, u_3\} \to 0\{u_0, u_1, u_3\},\ \{u_0, u_1, u_3\} \to 1\{u_2, u_3\},$$
$$\{u_2, u_3\} \to 0\{u_0, u_1, u_3\}, \{u_2, u_3\} \to 1\{u_2\}, \{u_2\} \to 0\{u_0, u_1\}, \{u_2\} \to 0\{\ \},$$
$$\{u_0, u_1\} \to 0\{u_0, u_1\},\ \text{and}\ \{u_0, u_1\} \to 1\{u_2, u_3\}, \tag{7}$$

where { } represents the empty set.

Given the extended relation set (7), we can easily construct the DFA for rule 76 as shown in Fig. 2.

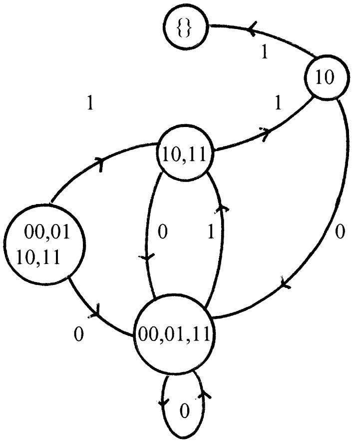


**Fig. 3.** The minimal representation of the DFA for rule 76

*Step 3.* Minimize the DFA to obtain the unique representation

The DFA obtained from Steps 1 and 2 may not be unique because some nodes can be further combined in light of these extended relations in order to reduce the size of the DFA. The minimization step is to find pairs of nodes where the arcs with the same values are directed to the same nodes, and then combine these pairs of nodes into single ones. Consider the extended relation set in (7) and the DFA shown in Fig. 2 for rule 76. The nodes $\{u_0, u_1, u_3\}$ and $\{u_0, u_1\}$ are directed to themselves given the output value as zero, while to $\{u_2, u_3\}$ given the output value as unity. The two nodes can thus be combined into $\{u_0, u_1, u_3\}$ with other extended relations remaining the same. A closer examination cannot find further combinations, and the resulting DFA as shown in Fig. 3 is the minimal representation of the DFA for rule 76, which is also unique.

## Appendix B
## Classification of transition rules into semi-lattices and trees

The 256 transition rules of the one-dimensional cellular automata with two neighbors ($r = 1$) and two cell values ($k = 0$ or 1) are classified into 12 categories according to rule types (trees, semi-lattices, or other) and structure classes (Classes 1 through 4): S1 stands for semi-lattice, Class 1 rules; S2 semi-lattice, Class 2; S3 semi-lattice, Class 3; S4 semi-lattice, Class 4; T1 tree, Class 1; T2 tree, Class 2; T3 tree, Class 3; T4 tree, Class 4; O1 other, Class 1; O2 other, Class 2; O3 other, Class 3; and O4 other, Class 4. The determination of the semi-lattice, tree, or other structural rules is based on the definition given in Sect. 3, according to the minimization algorithm as illustrated in Appendix A. The resulting classification is shown in Table 4.

**Table 4.** Classification of transition rules into semi-lattices and trees

| Rule number | Binary code | Classification | Rule number | Binary code | Classification |
|---|---|---|---|---|---|
| 0 | 00000000 | O1 | 55 | 00110111 | T1 |
| 1 | 00000001 | T2 | 56 | 00111000 | T2 |
| 2 | 00000010 | T2 | 57 | 00111001 | S1 |
| 3 | 00000011 | T2 | 58 | 00111010 | S1 |
| 4 | 00000100 | S2 | 59 | 00111011 | S2 |
| 5 | 00000101 | S2 | 60 | 00111100 | O3 |
| 6 | 00000110 | S2 | 61 | 00111101 | T2 |
| 7 | 00000111 | S1 | 62 | 00111110 | S1 |
| 8 | 00001000 | T1 | 63 | 00111111 | T1 |
| 9 | 00001001 | S4 | 64 | 01000000 | T1 |
| 10 | 00001010 | S2 | 65 | 01000001 | S4 |
| 11 | 00001011 | S2 | 66 | 01000010 | T2 |
| 12 | 00001100 | T2 | 67 | 01000011 | S2 |
| 13 | 00001101 | T1 | 68 | 01000100 | T2 |
| 14 | 00001110 | T2 | 69 | 01000101 | S1 |
| 15 | 00001111 | O2 | 70 | 01000110 | S1 |
| 16 | 00010000 | T2 | 71 | 01000111 | S2 |
| 17 | 00010001 | T2 | 72 | 01001000 | S1 |
| 18 | 00010010 | S3 | 73 | 01001001 | S3 |
| 19 | 00010011 | T1 | 74 | 01001010 | S2 |
| 20 | 00010100 | S2 | 75 | 01001011 | O3 |
| 21 | 00010101 | S1 | 76 | 01001100 | S2 |
| 22 | 00010110 | S3 | 77 | 01001101 | S1 |
| 23 | 00010111 | S1 | 78 | 01001110 | S1 |
| 24 | 00011000 | T2 | 79 | 01001111 | T1 |
| 25 | 00011001 | S2 | 80 | 01010000 | S2 |
| 26 | 00011010 | S3 | 81 | 01010001 | S2 |
| 27 | 00011011 | S2 | 82 | 01010010 | S3 |
| 28 | 00011100 | S1 | 83 | 01010011 | S2 |
| 29 | 00011101 | S2 | 84 | 01010100 | S2 |
| 30 | 00011110 | O3 | 85 | 01010101 | T2 |
| 31 | 00011111 | S1 | 86 | 01010110 | S3 |
| 32 | 00100000 | T1 | 87 | 01010111 | S1 |
| 33 | 00100001 | S2 | 88 | 01011000 | S2 |
| 34 | 00100010 | T2 | 89 | 01011001 | S3 |
| 35 | 00100011 | S2 | 90 | 01011010 | O3 |
| 36 | 00100100 | T2 | 91 | 01011011 | S2 |
| 37 | 00100101 | S2 | 92 | 01011100 | S1 |
| 38 | 00100110 | S2 | 93 | 01011101 | S1 |
| 39 | 00100111 | S2 | 94 | 01011110 | S1 |
| 40 | 00101000 | S1 | 95 | 01011111 | S1 |
| 41 | 00101001 | S4 | 96 | 01100000 | S1 |
| 42 | 00101010 | S2 | 97 | 01100001 | S4 |
| 43 | 00101011 | S2 | 98 | 01100010 | S2 |
| 44 | 00101100 | S2 | 99 | 01100011 | S1 |
| 45 | 00101101 | O3 | 100 | 01100100 | S2 |
| 46 | 00101110 | S2 | 101 | 01100101 | S3 |
| 47 | 00101111 | S2 | 102 | 01100110 | T3 |
| 48 | 00110000 | T2 | 103 | 01100111 | S2 |
| 49 | 00110001 | S2 | 104 | 01101000 | S1 |
| 50 | 00110010 | S1 | 105 | 01101001 | O3 |
| 51 | 00110011 | T2 | 106 | 01101010 | S3 |
| 52 | 00110100 | S2 | 107 | 01101011 | S4 |
| 53 | 00110101 | S2 | 108 | 01101100 | S2 |
| 54 | 00110110 | S3 | 109 | 01101101 | S3 |

**Table 4** (continued)

| Rule number | Binary code | Classification | Rule number | Binary code | Classification |
|---|---|---|---|---|---|
| 110 | 01101110 | S3 | 165 | 10100101 | O3 |
| 111 | 01101111 | S4 | 166 | 10100110 | S2 |
| 112 | 01110000 | T2 | 167 | 10100111 | S3 |
| 113 | 01110001 | S2 | 168 | 10101000 | S1 |
| 114 | 01110010 | S1 | 169 | 10101001 | S3 |
| 115 | 01110011 | S2 | 170 | 10101010 | T2 |
| 116 | 01110100 | S2 | 171 | 10101011 | S2 |
| 117 | 01110101 | S2 | 172 | 10101100 | S2 |
| 118 | 01110110 | S1 | 173 | 10101101 | S2 |
| 119 | 01110111 | T1 | 174 | 10101110 | S2 |
| 120 | 01111000 | O2 | 175 | 10101111 | S2 |
| 121 | 01111001 | S4 | 176 | 10110000 | T2 |
| 122 | 01111010 | S1 | 177 | 10110001 | S1 |
| 123 | 01111011 | S2 | 178 | 10110010 | S1 |
| 124 | 01111100 | S3 | 179 | 10110011 | S1 |
| 125 | 01111101 | S4 | 180 | 10110100 | O2 |
| 126 | 01111110 | T3 | 181 | 10110101 | S3 |
| 127 | 01111111 | T1 | 182 | 10110110 | S3 |
| 128 | 10000000 | T1 | 183 | 10110111 | S1 |
| 129 | 10000001 | T3 | 184 | 10111000 | S2 |
| 130 | 10000010 | S2 | 185 | 10111001 | S2 |
| 131 | 10000011 | S1 | 186 | 10111010 | S1 |
| 132 | 10000100 | S2 | 187 | 10111011 | T2 |
| 133 | 10000101 | S1 | 188 | 10111100 | S2 |
| 134 | 10000110 | S2 | 189 | 10111101 | T2 |
| 135 | 10000111 | O3 | 190 | 10111110 | S2 |
| 136 | 10001000 | S1 | 191 | 10111111 | T1 |
| 137 | 10001001 | S3 | 192 | 11000000 | T1 |
| 138 | 10001010 | S2 | 193 | 11000001 | S3 |
| 139 | 10001011 | S2 | 194 | 11000010 | T2 |
| 140 | 10001100 | S2 | 195 | 11000011 | O3 |
| 141 | 10001101 | S1 | 196 | 11000100 | S2 |
| 142 | 10001110 | S2 | 197 | 11000101 | S1 |
| 143 | 10001111 | T2 | 198 | 11000110 | S1 |
| 144 | 10010000 | S2 | 199 | 11000111 | S1 |
| 145 | 10010001 | S1 | 200 | 11001000 | T1 |
| 146 | 10010010 | S3 | 201 | 11001001 | T1 |
| 147 | 10010011 | S3 | 202 | 11001010 | S2 |
| 148 | 10010100 | S2 | 203 | 11001011 | S2 |
| 149 | 10010101 | S3 | 204 | 11001100 | S2 |
| 150 | 10010110 | O3 | 205 | 11001101 | T2 |
| 151 | 10010111 | S1 | 206 | 11001110 | S1 |
| 152 | 10011000 | S2 | 207 | 11001111 | T2 |
| 153 | 10011001 | T3 | 208 | 11010000 | S2 |
| 154 | 10011010 | S3 | 209 | 11010001 | S2 |
| 155 | 10011011 | S2 | 210 | 11010010 | O3 |
| 156 | 10011100 | S1 | 211 | 11010011 | S2 |
| 157 | 10011101 | S1 | 212 | 11010100 | S2 |
| 158 | 10011110 | S2 | 213 | 11010101 | S2 |
| 159 | 10011111 | S1 | 214 | 11010110 | S2 |
| 160 | 10100000 | S1 | 215 | 11010111 | S1 |
| 161 | 10100001 | S3 | 216 | 11011000 | S2 |
| 162 | 10100010 | S2 | 217 | 11011001 | S2 |
| 163 | 10100011 | S1 | 218 | 11011010 | S3 |
| 164 | 10100100 | S2 | 219 | 11011011 | T2 |

**Table 4** (continued)

| Rule number | Binary code | Classification | Rule number | Binary code | Classification |
|---|---|---|---|---|---|
| 220 | 11011100 | S1 | 238 | 11101110 | T1 |
| 221 | 11011101 | T2 | 239 | 11101111 | T1 |
| 222 | 11011110 | S1 | 240 | 11110000 | O2 |
| 223 | 11011111 | T1 | 241 | 11110001 | T2 |
| 224 | 11100000 | S1 | 242 | 11110010 | T1 |
| 225 | 11100001 | O3 | 243 | 11110011 | T2 |
| 226 | 11100010 | S2 | 244 | 11100100 | S2 |
| 227 | 11100011 | S2 | 245 | 11110101 | S2 |
| 228 | 11100100 | S2 | 246 | 11110110 | S1 |
| 229 | 11100101 | S2 | 247 | 11110111 | T1 |
| 230 | 11100110 | S2 | 248 | 11111000 | S1 |
| 231 | 11100111 | T2 | 249 | 11111001 | S1 |
| 232 | 11101000 | S1 | 250 | 11111010 | S1 |
| 233 | 11101001 | S1 | 251 | 11111011 | S1 |
| 234 | 11101010 | S2 | 252 | 11111100 | T1 |
| 235 | 11101011 | S1 | 253 | 11111101 | T1 |
| 236 | 11101100 | T2 | 254 | 11111110 | T1 |
| 237 | 11101101 | S1 | 255 | 11111111 | O1 |